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Periodic Positive Solutions for a Delay Nonlinear Differential Equation with Piecewise Constant Arguments

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Abstract—Using a fixed-point theorem in cone, existence criteria are proved for the periodic positive solutions of a delay differential equation with piecewise constant arguments. Several examples concerning biological models for the survival of red blood cells are given. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Differential equation, Piecewise constant argument, Periodic positive solution, Cone, Fixed-point theorem.

1. INTRODUCTION

Considerable attention has been given to delay differential equations with piecewise constant arguments by several authors including Cooke and Wiener [1], Shah and Wiener [2], and Aftabizadeh *et al.* [3]. This class of differential equations has useful applications in biomedical models of disease that have been developed by Busenerg and Cooke [4]. Studies of such equations were motivated by the fact that they represent a hybrid of discrete and continuous dynamical systems and combine the properties of both differential and differential-difference equation.

On the other hand, properties and solutions of delay differential equations with piecewise constant arguments and piecewise constant time delay have received considerable attention by several authors including Wiener [5], Cooke and Wiener [6], Wiener and Cooke [7], Wiener and

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Debnath [8,9], Gopalsamy et al. [10], Lin and Wang [11], Papaschinopoulos and Schinas [12], Huang [13], Shen and Stavroulakis [14], and Wiener and Heller [15].

Carvalho and Wiener [16] considered periodic solutions of the first-order differential equation with piecewise constant argument

$$x'(t) = ax(t)(1 - x([t])), \tag{1.1}$$

where a is constant and $[\cdot]$ is the greatest integer function. As mentioned by Cooke and Wiener [17], equation (1.1) may be treated as semidiscretization of the ordinary logistic equation and solutions of (1.1) exhibit a wide variety of properties of interest.

In this paper, we consider a more general equation than (1.1), in the form

$$x'(t) = a(t)x(t) + f(t, x([t])), x([t-1]), \dots, x([t-k]), x([t]), \tag{1.2}$$

and

$$x'(t) = a(t)x(t) - f(t, x([t])), x([t-1]), \dots, x([t-k]), x([t]), \tag{1.3}$$

where $a(t) \in C(R)$ and $f(t, x_0, \dots, x_{k+1})$ is a periodic function with a period ω, $f(t, x_0, \dots, x_k, x_{k+1}) \in C(R \times [0, \infty)^{k+2})$.

The main objective of this paper is to prove several existence criteria for ω-periodic positive solutions of equation (1.2) or (1.3) by using the fixed-point theorem in a cone due to Krasnosel'skii (see [18]).

We adopt the same definition for a solution of (1.2) or (1.3) as given by Aftabbizadeh et al. [3]. By a solution of (1.2) or (1.3), we mean a function $y(t)$ which is defined on the set $S = \{-k, -k+1, \dots, 0\} \cup [0, \infty)$ and this function $y(t)$ satisfies the following conditions:

(i) $y(t) = D_i$ $(i = 0, -1, \dots, -k)$, where D_i $(i = 0, -1, \dots, -k)$ is initial data;
(ii) $y(t)$ is continuous on $[0, \infty)$;
(iii) the derivative $y'(t)$ exists at each point $t \in [0, \infty)$, with the possible exception of the point $[t] \in [0, \infty)$, where one-sided derivative exists;
(iv) equation (1.2) or (1.3) is satisfied on each interval $[n, n+1)$ with integral endpoints.

We also state the fixed-point theorem due to Krasnosel'skii (see [18]) in the following form.

THEOREM 1.1. *Suppose K is a cone in Banach space X, Ω_1 and Ω_2 are two bounded open sets in X such that $\theta \in \Omega_1$, and $\bar{\Omega}_1 \subset \Omega_2$ and $\Phi : K \to K$ is a completely continuous operator. Further, suppose that any one of the following conditions is satisfied:*

(i) *for each $u \in K \cap \partial\Omega_1$, $\|\Phi u\| \le \|u\|$, and for each $u \in K \cap \partial\Omega_2$, $\|\Phi u\| \ge \|u\|$;*
(ii) *for each $u \in K \cap \partial\Omega_1$, $\|\Phi u\| \ge \|u\|$, and for each $u \in K \cap \partial\Omega_2$, $\|\Phi u\| \le \|u\|$;*

where $\partial\Omega_1$ and $\partial\Omega_2$ represent the boundary of Ω_1 and Ω_2, respectively. Then Φ has a fixed point $u_0 \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

The notation $x \to 0$ denotes that x_i $(i = 0, \dots, k, k+1)$ altogether tend to 0, and $x \to \infty$ denotes that all x_i $(i = 0, \dots, k+1)$ tend to infinity.

We use the following conditions. Let α_i $(i = 0, \dots, k+1)$ be nonnegative numbers and $\sum_{i=0}^{k+1} \alpha_i = 1$ such that

(H$_1$)

$$\overline{\lim}_{x \to 0^+} \max_{0 \le t \le \omega} \frac{f(t, x_0, \dots, x_k, x_{k+1})}{\prod_{i=0}^{k+1} |x_i|^{\alpha_i}} = 0$$

and

$$\underline{\lim}_{x \to \infty} \min_{0 \le t \le \omega} \frac{f(t, x_0, \dots, x_k, x_{k+1})}{\prod_{i=0}^{k+1} |x_i|^{\alpha_i}} = \infty;$$

(H_2)

$$\underline{\lim}_{x\to 0^+} \min_{0\le t\le\omega} \frac{f(t,x_0,\dots,x_k,x_{k+1})}{\prod_{i=0}^{k+1}|x_i|^{\alpha_i}} = \infty$$

and

$$\overline{\lim}_{x\to\infty} \max_{0\le t\le\omega} \frac{f(t,x_0,\dots,x_k,x_{k+1})}{\prod_{i=0}^{k+1}|x_i|^{\alpha_i}} = 0.$$

2. MAIN RESULTS

THEOREM 2.1. *If condition (H_1) is satisfied, then equation (1.2) has an ω-periodic positive solution.*

THEOREM 2.2. *If condition (H_2) is satisfied, then equation (1.2) has an ω-periodic positive solution.*

THEOREM 2.3. *If condition (H_1) is fulfilled, then equation (1.3) has an ω-periodic positive solution.*

THEOREM 2.4. *If condition (H_2) is satisfied, then equation (1.3) has an ω-periodic positive solution.*

We only give proofs of Theorem 2.1 and 2.2. Theorems 2.3 and 2.4 can be proved similarly.

Let X be the Banach space of all functions $x = x(t)$ which are defined on $\{-k,\dots,0\}\cup[0,\infty)$ and continuous on $[0,\infty)$ such that $x(t+\omega) = x(t)$ for $t\in\{-k,\dots,0\}\cup[0,\infty)$, endowed with the norm $\|x\| = \max_{0<t<\omega}|x(t)|$. If

$$K = \{x\in X,\ x(t)\ge\sigma\|x\|\},$$

where $\sigma = \exp(-\int_0^\omega a(\nu)\,d\nu)$, and $a(x)$ is defined before, then K is a cone in X. We next set

$$G(t,s) = \frac{\exp\left(\int_t^s a(\nu)\,d\nu\right)}{\exp\left(\int_0^\omega a(\nu)\,d\nu - 1\right)}. \tag{2.1}$$

It is easy to see that $x(t)$ is an ω-periodic positive solution of (1.2) if and only if $x(t)$ is an ω-periodic positive solution of the following equation:

$$x(t) = \int_t^{t+\omega} G(t,s) f(s,x[s]),x([s-1]),\dots,x([s-k]),x([s])\,ds. \tag{2.2}$$

If we define the mapping $\Phi : X\to X$ by

$$(\Phi x)(t) = \begin{cases} \displaystyle\int_t^{t+\omega} G(t,s) f(s,x[s]),x([s-1]),\dots,x([s-k]),x([s])\,ds, & t\ge 0,\\ (\Phi x)(i-i\omega), \quad t=i \quad (i=0,-1,\dots,-k), & \omega<1, \end{cases} \tag{2.3}$$

then Φ is completely continuous, and for $x\in K$,

$$\begin{aligned} \|\Phi x\| &\le G(t,t+\omega)\int_0^\omega f(s,x[s]),x([s-1]),\dots,x([s-k]),x([s])\,ds,\\ &= G(0+\omega)\int_0^\omega f(s,x[s]),x([s-1]),\dots,x([s-k]),x([s])\,ds. \end{aligned} \tag{2.4}$$

Thus,

$$
\begin{aligned}
(\Phi x)(t) &\geq G(t,t)\int_0^\omega f(s,x[s]),x([s-1]),\dots,x([s-k]),x([s])\,ds,\\
&= G(0,0)\int_0^\omega f(s,x[s]),x([s-1]),\dots,x([s-k]),x([s])\,ds, \qquad (2.5)\\
&\geq \frac{G(0,0)}{G(0,\omega)}\|\Phi x\| = \sigma\|\Phi x\|, \qquad t\geq 0.
\end{aligned}
$$

Noting that $i-i\omega\geq 0$, $(\omega<1)$ for $i=0,-1,\dots,-k$, by (2.3) and (2.5) and the definition of K, we have

$$(\Phi x)(t) = (\Phi x)(i-i\omega)\geq \sigma\|x\|, \qquad t=i, \quad (i=0,-1,\dots,-k). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\Phi x\in K, \qquad \text{that is,} \qquad \Phi K\subset K.$$

PROOF OF THEOREM 2.1. First of all, we set a positive number ε such that $G(0,\omega)\omega\varepsilon<1$, it follows from (H_1) that

$$\overline{\lim}_{x\to 0^+}\max_{0\leq t\leq\omega}\frac{f(t,x_0,\dots,x_k,x_{k+1})}{\prod_{i=0}^{k+1}|x_i|^{\alpha_i}} = 0.$$

Then there exists a $\rho_1>0$ such that

$$f(t,x_0,\dots,x_k,x_{k+1})\leq\varepsilon\prod_{i=0}^{k+1}|x_i|^{\alpha_i}, \qquad |x_i|\leq\rho_1, \quad (i=0,1,\dots,k+1). \tag{2.7}$$

In view of (2.4) and (2.7), for each $x\in K$ and $\|x\|=\rho_1$, it turns out that

$$
\begin{aligned}
\|\Phi x\| \max_{0\leq t\leq\omega}|(\Phi x)(t)| &\leq G(0,\omega)\int_0^\omega f(s,x[s]),x([s-1]),\dots,x([s-k]),x([s])\,ds,\\
&\leq G(0,\omega)\varepsilon\int_0^\omega |x(s)|^{\alpha_{k+1}}\prod_{i=0}^{k} x|([s-i])|^{\alpha_i}\,ds, \qquad (2.8)\\
&\leq \frac{G(0,0)}{G(0,\omega)}\varepsilon\omega\|x\| < \|x\|.
\end{aligned}
$$

In other words, for each $x\in K\cap\partial\Omega_1$, $\|\Phi x\|<\|x\|$, where $\Omega_1=\{x\in X\|x\|<\rho_1\}$. Let M be a positive number such that $G(0,0)\omega\sigma M>1$. It follows from (H_1) that

$$\underline{\lim}_{x\to\infty}\min_{0\leq t\leq\omega}\frac{f(t,x_0,\dots,x_k,x_{k+1})}{\prod_{i=0}^{k+1}|x_i|^{\alpha_i}} = \infty.$$

Then there exist $\rho_2>0$ and $\rho_2>\rho_1$, such that

$$f(t,x_0,\dots,x_k,x_{k+1})\geq M\prod_{i=0}^{k+1}|x_i|^{\alpha_i}, \qquad |x_i|\geq\frac{\rho_2}{\sigma}, \quad (i=0,1,\dots,k+1). \tag{2.9}$$

Using (2.3) and (2.9), for each $x\in K$ and $\|x\|=\rho_2$, we have

$$
\begin{aligned}
\|\Phi x\| = \max_{0\leq t\leq\omega}|(\Phi x)(t)| &\geq G(0,0)\int_0^\omega f(s,x([s])),x([s-1]),\dots,x([s-k]),x([s]),ds\\
&\geq G(0,0)M\int_0^\omega |x(s)|^{\alpha_{k+1}}\prod_{i=0}^{k}|x([s-i])|^{\alpha_i}\,ds \qquad (2.10)\\
&\geq G(0,0)M\sigma\|x\| > \|x\| = \rho_2.
\end{aligned}
$$

That is, for each $x \in K \cap \partial\Omega_2$, $\|\Phi x\| > \|x\|$, where $\Omega_2 = \{x \in X \|x\| < \rho_2\}$. It follows from relations (2.8), (2.10), and Theorem 1.1 with $\Omega_1 \subset \Omega_2$ that the mapping Φ has a fixed point $x_0 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. In view of definitions of Ω_1 and K, we have $x(t) \geq \sigma\|x\| \geq \sigma\rho_1 > 0$. Thus, $x(t)$ is an ω-periodic positive solution of (1.2). The proof of Theorem 2.1 is complete.

PROOF OF THEOREM 2.2.. Let M be a positive number such that $G(0,0)\omega\sigma M > 1$. It follows from (H_2) that

$$\underline{\lim}_{x\to 0} \min_{0\leq t\leq\omega} \frac{f(t, x_0, \dots, x_k, x_{k+1})}{\prod\limits_{i=0}^{k+1} |x_i|^{\alpha_i}} = \infty.$$

Then there exists a $\rho_1 > 0$ such that

$$f(t, x_0, \dots, x_k, x_{k+1}) \geq M \prod_{i=0}^{k+1} |x_i|^{\alpha_i}, \qquad |x_i| \leq \rho_1, \quad (i = 0, 1, \dots, k+1). \tag{2.11}$$

By (2.3) and (2.11), for each $x \in K$ and $\|x\| = \rho_1$, we have the following inequality:

$$\begin{aligned}\|\Phi x\| &= \max_{0\leq t\leq\omega} |(\Phi x)(t)| \geq G(0,0) \int_0^\omega f(s, x([s])), x([s-1]), \dots, x([s-k]), x([s])\, ds \\ &\geq G(0,0)M \int_0^\omega |x(s)|^{\alpha_{k+1}} \prod_{i=0}^{k} |x([s-i])|^{\alpha_i}\, ds \\ &\geq G(0,0)M\omega\sigma\|x\| > \|x\| = \rho_1.\end{aligned} \tag{2.12}$$

That is, for each $x \in K \cap \partial\Omega_1$, $\|\Phi x\| > \|x\|$, where $\Omega_1 = \{x \in X\|x\| < \rho_1\}$. Let $\varepsilon > 0$ such that $G(0,\omega)\omega\varepsilon > 1$. It follows from ($\mathrm{H}_2$) that

$$\overline{\lim}_{x\to\infty} \max_{0\leq t\leq\omega} \frac{f(t, x_0, \dots, x_k, x_{k+1})}{\prod\limits_{i=0}^{k+1} |x_i|^{\alpha_i}} = 0.$$

Hence, there exist $\rho_2 > \rho_1 > 0$ such that

$$f(t, x_0, \dots, x_k, x_{k+1}) \leq \varepsilon \prod_{i=0}^{k+1} |x_i|^{\alpha_i}, \qquad |x_i| \geq \frac{\rho_2}{\sigma}, \quad (i = 0, 1, \dots, k+1). \tag{2.13}$$

By (2.4) and (2.13), for each $x \in K$ and $\|x\| = \rho_2$, we have the following inequality:

$$\begin{aligned}\|\Phi x\| &= \max_{0\leq t\leq\omega} |(\Phi x)(t)| \leq G(0,\omega) \int_0^\omega f(s, x([s])), x([s-1]), \dots, x([s-k]), x([s])\, ds \\ &\leq G(0,\omega)\varepsilon \int_0^\omega |x(s)|^{\alpha_{k+1}} \prod_{i=0}^{k} |x([s-i])|^{\alpha_i}\, ds \\ &\leq G(0,\omega)\varepsilon\omega\|x\| < \|x\|.\end{aligned} \tag{2.14}$$

That is, for each $x \in K \cap \partial\Omega_2$, $\|\Phi x\| < \|x\|$, where $\Omega_2 = \{x \in X\|x\| < \rho_2\}$. It follows from relations (2.12), (2.14), and Theorem 1.1 with $\Omega_1 \subset \Omega_2$ that the mapping Φ has a fixed point $x_0 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. In view of definitions of Ω_1 and K, we have $x(t) \geq \sigma\|x\| \geq \sigma\rho_1 > 0$. Thus, $x(t)$ is an ω-periodic positive solution of (1.2). The proof of Theorem 2.2 is complete.

3. EXAMPLES

EXAMPLE 1. The model for the survival of red blood cells is given in [19–21] of the form

$$x'(t) = -a(t)x(t) + b(t)\exp[x(t-\tau(t))], \tag{3.1}$$

where $x(t)$ denotes the number of red blood cells at time t, $a(t)$ is the probability of death of a red cell, $b(t)$ is related to the production of red blood cells, and $\tau(t)$ is the time required to produce a red blood cell. Following Kocic and Ladas [21], we consider a semidiscrete model, for the survival of red blood cells in an animal which satisfies the equation

$$x'(t) = -a(t)x(t) + b(t)\exp(x([t-k])), \qquad t \geq 0, \tag{3.2}$$

where $a(t)$ and $b(t)$ belong to $C(R)$, $a(t+\omega) = a(t)$ and $b(t+\omega) = b(t)$. Let $f(t,x) = b(t)\exp(-x)$. We see that

$$\underline{\lim}_{x\to 0^+} \min_{0\leq t\leq\omega} \frac{f(t,x)}{|x|} = \infty \qquad \text{and} \qquad \overline{\lim}_{x\to\infty} \max_{0\leq t\leq\omega} \frac{f(t,x)}{|x|} = 0.$$

Using Theorem 2.2 gives that (3.2) has an ω-periodic positive solution.

EXAMPLE 2. The model for the generation of red blood cells in an animal is described in [22,23] of the form

$$x'(t) + a(t)x(t) = \frac{b(t)}{1+(x(t-\tau(t)))^n}. \tag{3.3}$$

We follow Kocic and Ladas [21] to study solutions of discrete analogue of (3.3).

We consider a semidiscrete model for the generating of red blood cells in an animal that satisfies the equation

$$x'(t) = -a(t)x(t) + \frac{b(t)}{1+(x([t-k]))^n}, \qquad t \geq 0, \tag{3.4}$$

where n is a positive integer, $a(t)$ and $b(t)$ belong to $C(R)$, $a(t+\omega) = a(t)$ and $b(t+\omega) = b(t)$. Suppose $f(t,x) = b(t)/(1+|x|^n)$. We obtain

$$\underline{\lim}_{x\to 0^+} \min_{0\leq t\leq\omega} \frac{f(t,x)}{|x|} = \infty \qquad \text{and} \qquad \overline{\lim}_{x\to\infty} \max_{0\leq t\leq\omega} \frac{f(t,x)}{|x|} = 0.$$

Using Theorem 2.2 gives that (3.2) has an ω-periodic positive solution.

EXAMPLE 3. Making reference to [24–26], we state a model equation for generation of a biological model in the form

$$x'(t) = x(t)\left[a(t) - \sum_{i=1}^{n} b_i(t)(x(t-\tau_i(t)))\right]. \tag{3.5}$$

We consider a semidiscrete biological model which satisfies the equation

$$x'(t) = x(t)\left(a(t) - \prod_{i=0}^{k} \frac{x([t-i])}{b(t)}\right), \tag{3.6}$$

where k is a positive integer, $a(t)$ and $b(t)$ belong to $C(R)$, $a(t+\omega) = a(t)$ and $b(t+\omega) = b(t)$. Let $f(t,x_0,\dots,x_k,x_{k+1}) = \prod_{i=0}^{k+1} |x_i|/(b(t))^k$, and set $\alpha_1 = 0$ $(i = 0,1,\dots,k)$, $\alpha_{k+1} = 1$. Then $\sum_{i=0}^{k+1} \alpha_i = 1$. It is easy to see that condition (H_1) is satisfied. Using Theorem 2.3 leads to the fact that (3.5) has an ω-periodic positive solution.

EXAMPLE 4. We consider a semidiscrete biological model which satisfies the equation

$$x'(t) = x(t)\left(a(t) - \sum_{i=0}^{k} b_i(t)x([t-i])\right), \tag{3.7}$$

where k is a positive integer, $a(t)$ and $b(t)$ belong to $C(R)$, $a(t+\omega) = a(t)$ and $b_i(t+\omega) = b_i(t)$, $(i = 0,1,\dots,k)$. By Theorem 2.3, we know that (3.7) has an ω-periodic positive solution.

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